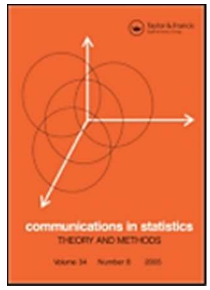

On Integral Functionals of a Density

Journal:	*Communications in Statistics – Theory and Methods*
Manuscript ID:	LSTA-2014-0585
Manuscript Type:	Original Paper
Date Submitted by the Author:	14-Jul-2014
Complete List of Authors:	Sokhadze, Grigol; Applied Mathematical Institute, Theory of Probability and Mathematical Statistics
Keywords:	consistency, Plug-in-estimator, central limit theorem, information, entropy
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On Integral Functionals of a Density

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Abstract

Estimation of a nonlinear integral functional of probability distribution density and its derivatives is studied. The truncated plug-in-estimator is taken for the estimation. The integrand function can be unlimited, but it cannot exceed polynomial growth. Consistency of the estimator is proved and the convergence order is established. Aversion of the central limit theorem is proved. As an example an extended Fisher information integral and generalized Shannon's entropy functional are considered.

2010 Mathematics Subject Classification. Primary 62G07, 62G20; Secondary 62E20.

Key words and phrases. Plug-in-estimator, integral functional, consistency, central limit theorem, information, entropy.

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1 Introduction

Consider the functional of following type

$$I(f) = \int_{-\infty}^{\infty} \varphi\big(x, f(x), f'(x), \dots, f^{(m)}(x)\big)\, dx, \tag{1}$$

where φ is a smooth function of $m+2$ variables; $f(x)$ is an unknown probability distribution density of a random variable X; $f^{(k)}(x)$, $k = 0, 1, \dots, m$ is a derivative of the function $f(x)$ of the order k, $f^{(0)}(x) \stackrel{def}{=} f(x)$. Let $X_1, X_2, \dots, X_n$ be a sample of independent identically distributed random variables each of which has a distribution coinciding with the distribution of X. The problem of statistical estimation of the functional $I(f)$ will be studied on the basis of this sample using the truncated plug-in-estimator:

$$I(\widehat{f}_n, s_n) = \int_{-s_n}^{s_n} \varphi\big(x, \widehat{f}_n(x), \widehat{f}'_n(x), \dots, \widehat{f}_n^{(m)}(x)\big)\, dx, \tag{2}$$

where $\widehat{f}_n(x)$ is the estimator of the density $f(x)$, and $\widehat{f}_n^{(k)}(x)$, $k = 0, 1, \dots, m$, is the derivative of the function $\widehat{f}_n(x)$ order k, $\widehat{f}_n^{(0)}(x) \stackrel{def}{=} \widehat{f}_n(x)$. As an estimator of $f(x)$ and its derivatives we give the kernel probability density estimator obtained by Rosenblatt–Parzen (Rosenblatt (1956), Parzen (1962), Bhattacharya (1967), Schuster (1969), Nadaraya (1988)). They have the following form

$$\widehat{f}_n^{(k)}(x) = \frac{1}{nh_n^{k+1}} \sum_{i=1}^{n} K^{(k)}\Big(\frac{x - X_i}{h_n}\Big), \quad k = 0, 1, \dots m. \tag{3}$$

The problem of estimation of the integral functional of form (1) and its particular varieties has been studied by a number of authors. Results have been obtained where consistency and other asymptotic properties are established. These properties were used in some quite interesting studies in order to define properties of specific integral functionals of application significance (Levit (1978), Hall and Marron (1987), Bickel and Ritov (1988), Birge and Massart (1995), Laurent (1996), Mason (2003), Gine and Mason (2008), Mason, Nadaraya and Sokhadze (2010)).

The results obtained in the above-mentioned works cannot, however, be applied to some quite important cases. This mainly refers to functional in which the function φ is unlimited. In, particular, it concerns the Fisher information estimator and Shannon's entropy

$$I_F(f) = \int_{-\infty}^{\infty} \frac{(f'(x))^2}{f(x)}\,dx, \quad I_S(f) = \int_{-\infty}^{\infty} f(x)\log f(x)\,dx. \tag{4}$$

Integral functionals of form (4) have been the subject of separate studies. We should particularly mention Bhattacharya's work (1967), where an efficient approach to the estimation of these integrals is given. As an extension of this work the paper by Dmitriev and Tarasenko (1973) can be mentioned.

The aim of the given article is to study the asymptotic properties of type (2) functional as an estimator of functional (1) so that it would also cover the case of integral functional of type (4).

2 Preliminaries

Introduce the notation and conditions that we will need in the forthcoming.

The following conditions are assumed to be satisfied for the function φ:

Assumption (φ1). $\varphi(x, x_0, \ldots, x_m)$ is a function of $m+2$ variables, which has an open definition domain D_φ, takes real values, is continuous with respect to the set of variables and has continuous partial derivatives up to the second order inclusively, with respect to the variables $x_0, \ldots, x_m$.

For simplicity we denote partial derivatives of the function φ in the following way:

$$\frac{\partial\varphi(x, x_0, x_1, \ldots, x_m)}{\partial x_i} \stackrel{def}{=} \varphi_{(i)}(x, x_0, x_1, \ldots, x_m) \stackrel{def}{=} \varphi_{(i)}, \quad i = 0, 1, \ldots, m,$$
$$\frac{\partial^2\varphi(x, x_0, \ldots, x_m)}{\partial x_i x_j} \stackrel{def}{=} \varphi_{(ij)}(x, x_0, \ldots, x_m) \stackrel{def}{=} \varphi_{(ij)}, \quad i, j = 0, 1, \ldots, m.$$

Assumption (φ2). The derivatives of the function φ satisfy growth conditions: constants $\alpha_{0i} \in R$, $\beta_{0ij} \in R$ and $C_\varphi > 0$, $\alpha_i \geq 0$, $\beta_{ij} \geq 0$, $\alpha_{1i} \geq 0, \ldots, \alpha_{mi} \geq 0$, $\beta_{1ij} \geq 0, \ldots, \beta_{mij} \geq 0$, exist, such that for any admissible values of the arguments and for each $i, j = 0, 1, \dots, m$, we have

$$\left|\varphi_{(i)}(x, x_0, x_1, \dots, x_m)\right| \le C_\varphi |x|^{\alpha_i} |x_0|^{\alpha_{0i}} |x_1|^{\alpha_{1i}} \cdots |x_m|^{\alpha_{mi}}, \tag{5}$$

$$\left|\varphi_{(ij)}(x, x_0, x_1, \dots, x_m)\right| \le C_\varphi |x|^{\beta_{ij}} |x_0|^{\beta_{0ij}} |x_1|^{\beta_{1ij}} \dots |x_m|^{\beta_{mij}}. \tag{6}$$

Note that in inequalities (5) and (6) constants α_{0i} and β_{0ij}, $i, j = 0, 1, \dots, m$, can be both positive or negative (or equal to 0).

Assumption (φ3). Integral $\int\limits_{-\infty}^{\infty} \varphi(x, f(x), f'(x), \dots, f^{(m)}(x))\, dx$ exists.

Let X be a random variable with probability distribution density $f(x)$. Assume that the following conditions are satisfied for it:

Assumption (f1). The density function $f(x)$ has continuous derivatives up to the order $m \ge 0$ inclusive.

Assumption (f2). For a $C_f > 0$, we have $\sup\limits_{x \in \mathbb{R}} |f^{(k)}(x)| \le C_f < \infty$, $k = 0, 1, \dots, m$.

Assumption (f3). For every $k = 1, \dots, m$, $f^{(k)} \in L_1(\mathbb{R})$.

Assumption (f4). Exists strictly growing function $H(x)$ such that

$$\sup_{|y| \le x} \frac{1}{f(y)} \le H(x). \tag{7}$$

Without loss of generality, assume that $H(x) \ge x$. Introduce the function

$$V(\alpha; x) \stackrel{def}{=} \begin{cases} C_f^\alpha, & \text{if } \alpha \ge 0, \\ (H(x))^{-\alpha}, & \text{if } \alpha < 0. \end{cases}$$

Consider a real-valued nonnegative function $K(x)$ and assume that the following conditions hold:

Assumption (k1). $\int\limits_{-\infty}^{\infty} K(x)\, dx = 1$.

Assumption (k2). $K(x)$ has continuous derivatives up to the order m inclusive.

Assumption (k3). For a $C_K > 0$, $|K^{(k)}(x)| \le C_K < \infty$, $k = 0, 1, \dots, m$.

Assumption $(\boldsymbol{k}4)$. The function $K(x)$ has a compact support which lies within the interval $[-\Bbbk, \Bbbk]$.

$X_1, X_2, \ldots, X_n$ is a sample of independent identically distributed random variables each of which having a distribution coinciding with the distribution of X. Apply kernel estimator (3) to the unknown function $f^{(k)}(x)$, $k = 0, 1, \ldots, m$. For the sequence $\{h_n\}_{n=1}^{\infty}$ the following condition is satisfied:

Assumption $(\boldsymbol{h})$. h_n, $n = 1, 2, \ldots$, is a sequence of positive numbers monotonically converging to 0, such that $h_n \geq \frac{c \log n}{n}$ for a $c > 0$.

It is known (Dony, Einmahl and Mason (2006)), that under the conditions $(\boldsymbol{f}1)$–$(\boldsymbol{f}3)$, $(\boldsymbol{k}1)$–$(\boldsymbol{k}4)$ and $(\boldsymbol{h})$, for every $k = 0, 1, \ldots, m$, with probability equal to 1, we have

$$\sup_{x \in \mathbb{R}} \big| \widehat{f}_n^{(k)}(x) - E\widehat{f}_n^{(k)}(x) \big| = O\left(\frac{\sqrt{|\log h_n| \vee \log\log n}}{\sqrt{n}\, h_n^{0,5+k}} \right). \tag{8}$$

Let $f_n^{(k)}(x) = E\widehat{f}_n^{(k)}(x)$. We have an equality

$$f_n^{(k)}(x) = E\widehat{f}_n^{(k)}(x) = \frac{1}{h_n^{k+1}} \int\limits_{-\infty}^{\infty} K^{(k)}\Big(\frac{x-t}{h_n}\Big) f(t)\, du.$$

Applying the formula of integration by parts several times we obtain

$$f_n^{(k)}(x) = \int\limits_{-\infty}^{\infty} K(u) f^{(k)}(x - uh_n)\, du. \tag{9}$$

Which together with the continuity of the functions $f^{(k)}(x)$, results in a point wise convergence $f_n^{(k)}(x) \to f^{(k)}(x)$, $x \in R$, for any $k = 0, 1, \ldots, m$ (see Parzen (1962), Theorem 1A). This, in its turn, implies the convergence

$$\varphi\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big) \to \varphi\big(x, f(x), \ldots, f^{(m)}(x)\big), \quad x \in R \ \text{ as } \ n \to \infty.$$

As it has been already mentioned above, a truncated integral functional (2) is taken as an estimator of functional (1).

The problem is to choose h_n and s_n so that $s_n \uparrow \infty$ as $n \to \infty$ and the convergence of $I(\widehat{f}_n)$ to $I(f)$ should be ensured with probability 1.

Introduce the following notation

$$I(f_n, s_n) \stackrel{def}{=} \int_{-s_n}^{s_n} \varphi\big(x, f_n(x), f_n'(x), \ldots, f_n^{(m)}(x)\big)\, dx,$$

$$I_1 \stackrel{def}{=} \Bigg| \int_{|x|\geq s_n} \varphi\big(x, f(x), f'(x), \ldots, f^{(m)}(x)\big)\, dx \Bigg|,$$

$$I_2 \stackrel{def}{=} \Bigg| \int_{-s_n}^{s_n} \Big\{ \varphi\big(x, f(x), \ldots, f^{(m)}(x)\big) - \varphi\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big) \Big\}\, dx \Bigg|,$$

$$r_n^{ij}(x) \stackrel{def}{=} \int_{-s_n}^{s_n} \big|\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big|\, \big|\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big| dx,$$

$$R_n \stackrel{def}{=} \frac{1}{2} \sum_{i,j=0}^{m} \int_{-s_n}^{s_n} \varphi_{(ij)}(\widetilde{y}_m(x)) \Big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\Big) \big(\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big)\, dx,$$

where $\widetilde{y}_m(x)$ is a point on the line connecting the points (defined below) $(x, f_n(x), \ldots, f_n^{(m)}(x))$ and $(x, \widehat{f}_n(x), \ldots, \widehat{f}_n^{(m)}(x))$.

Introduce the notation

$$\tau = \max_{0\leq i\leq m} \{1 + \alpha_i + |\alpha_{0i}|\}, \quad \sigma = \max_{0\leq i,\, j\leq m} \{\beta_{ij} + |\beta_{0ij}|\}, \quad \rho = \max\{\tau, \sigma\},$$

$$\theta = \min_{i,j} \{\alpha_i, \alpha_{0i}, \beta_{ij}, \beta_{0ij}\}, \quad U(x;\theta) = \begin{cases} x, & \text{if } \theta \geq 0, \\ H^{-1}(x), & \text{if } \theta < 0. \end{cases}$$

3 Results

Lemma 3.1. *Let the conditions* **(f1)**–**(f3)**, **(k1)**–**(k4)**, **(h)** *and* **(φ1)**–**(φ3)** *be satisfied.*

If $\theta \geq 0$ and s_n and h_n are selected so that

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i + 1 + \alpha_{0i}} \to 0 \;\text{ as }\; n \to \infty, \tag{10'}$$

then $I_2 \to 0$ as $n \to \infty$.

If $\theta < 0$, then under the conditions $(\boldsymbol{f1})$–$(\boldsymbol{f4})$, $(\boldsymbol{k1})$–$(\boldsymbol{k4})$, $(\boldsymbol{h})$, $(\boldsymbol{\varphi 1})$–$(\boldsymbol{\varphi 3})$ and s_n chosen h_n so that

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1} V(\alpha_{0i}; s_n) \to 0 \ \text{ as } \ n \to \infty, \tag{10''}$$

we have $I_2 \to 0$ as $n \to \infty$.

Remark 3.1. This lemma implies that under the conditions $(\boldsymbol{f1})$–$(\boldsymbol{f3})$, $(\boldsymbol{k1})$–$(\boldsymbol{k4})$, $(\boldsymbol{h})$ and $(\boldsymbol{\varphi 1})$–$(\boldsymbol{\varphi 3})$ the convergence $I_2 \to 0$ as $n \to \infty$ can take place in the following cases: if $\theta \geq 0$, then s_n and h_n should be selected so that we would have $(10')$, while if $\theta < 0$, then if an additional condition is satisfied $(\boldsymbol{f4})$, s_n and h_n must be chosen so that $(10'')$ would take place. It can be easily seen that in the last case $(10'')$ can be expressed as

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1+\alpha'_{0i}} H_{0i}^{\alpha''}(s_n) \to 0 \ \text{ for } \ n \to \infty,$$

where α'_{0i} denotes all nonnegative numbers from the numbers α_{0i}, and α''_{0i} are only negative ones out of the numbers α_{0i}, $i = 0, 1, \ldots, m$.

Lemma 3.2. *Let the conditions* $(\boldsymbol{f1})$–$(\boldsymbol{f3})$, $(\boldsymbol{k1})$–$(\boldsymbol{k4})$, $(\boldsymbol{h})$ *be satisfied. Then*

$$r_n^{ij}(x) = O\Big(\frac{\log n}{nh_n^{2m+1}}\Big). \tag{11}$$

Lemma 3.3. *If the conditions* $(\boldsymbol{f1})$–$(\boldsymbol{f4})$, $(\boldsymbol{k1})$–$(\boldsymbol{k4})$, $(\boldsymbol{h})$ *and* $(\boldsymbol{\varphi 1})$–$(\boldsymbol{\varphi 3})$ *hold, then*

$$R_n = O\Big(\frac{d_m(s_n)\log n}{nh_n^{2m+1}}\Big), \tag{12}$$

where

$$d_m(s_n) = \sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}; s_n). \tag{13}$$

Remark 3.2. This lemma results in the following estimators: if $\theta \geq 0$, then condition $(\boldsymbol{f4})$ is redundant and (13) takes the form

$$d_m(s_n) = \sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}+\beta_{0ij}},$$

and if $\theta < 0$ then if condition $(\boldsymbol{f4})$ is also satisfied, (13) takes the form

$$d_m(s_n) = \sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}+\beta'_{0ij}} H^{\beta''_{0ij}}(s_n),$$

where β'_{0ij} denotes nonnegative numbers out of the numbers β_{0ij}, and β''_{0ij} are negative ones out of the numbers β_{0ij}, $i = 0, 1, \dots, m$.

Theorem 3.1. *Let the conditions* $(\boldsymbol{f1})$–$(\boldsymbol{f3})$, $(\boldsymbol{k1})$–$(\boldsymbol{k4})$, $(\boldsymbol{h})$ *and* $(\boldsymbol{\varphi1})$–$(\boldsymbol{\varphi3})$ *hold and for* h_n *we have*

(A) $\dfrac{\log n}{n h_n^{2m+2}} \to \infty$;

(B) $\dfrac{\log n}{n h_n^{2m+1+\delta}} \to 0$ *for any* $0 \le \delta < 1$.

Let also $s_n = U(h_n^{-\frac{1}{1+\rho}}; \theta)$. *Then for the convergence* $(\widehat{f}_n, s_n) \to I(f)$ *with probability* 1, *it is sufficient that the condition:*

(i) $\theta \ge 0$ *is satisfied*

or

(ii) *if* $\theta < 0$, *then, in addition to the above-listed conditions, condition* $(\boldsymbol{f4})$ *should also hold.*

Remark 3.3. This theorem extends the known results on the consistency of the plug-in-estimator for particular integral functional of probability distribution density given in [9, 10, 11, 13, 14].

Theorem 3.2. *Let the conditions of Theorem* 3.1 *hold,* $f \in C^{2m}(R)$ *and the sequence of positive numbers* h_n *monotonically converge to zero, so that conditions* (A) *and* (B) *of Theorem* 3.1 *are satisfied. Hence if* $s_n = U(h_n^{-\frac{1}{1+\rho}}; \theta)$, *then*

$$\sqrt{n}\Big\{I(\widehat{f}_n, s_n) - I(f_n)\Big\} \xrightarrow{d} N\big(0, \sigma^2(f)\big).$$

4 Proofs

Proof of Lemma 3.1. We have

$$I_2 \le \int_{-s_n}^{s_n} \left| \sum_{i=0}^{m} \varphi_{(i)}\big(x, \widetilde{f}_n(x), \dots, \widetilde{f}_n^{(m)}(x)\big)\big(f^{(i)}(x) - f_n^{(i)}(x)\big) \right| dx,$$

where $(x, \widetilde{f}_n(x), \dots, \widetilde{f}_n^{(m)}(x))$ is a point on an interval connecting the points $(x, f(x), \dots, f^{(m)}(x))$ and $(x, f_n(x), \dots, f_n^{(m)}(x))$. Here

$$\widetilde{f}_n^{(i)}(x) = f^{(i)}(x) + \theta(f_n^{(i)}(x) - f^{(i)}(x)), \;\; 0 \le \theta \le 1, \;\; i = 0, 1, \dots, m.$$

Then by virtue of condition **(φ2)**

$$I_2 \le \int_{-s_n}^{s_n} \sum_{i=0}^{m} |x|^{\alpha_i} \big|\widetilde{f}_n(x)\big|^{\alpha_{0i}} \cdots |\widetilde{f}_n^{(m)}(x)|^{\alpha_{mi}} \big|f^{(i)}(x) - f_n^{(i)}(x)\big|\, dx. \qquad (14)$$

Since according to **(f2)** we have $\sup_{x\in\mathbb{R}} |f^{(k)}(x)| \le C_f < \infty$, $k = 1, \dots, m$, then (9) results in

$$\sup_{x\in\mathbb{R}} |f_n^{(k)}(x)| \le C_f < \infty, \;\; k = 0, 1, \dots, m.$$

Hence we can estimate $\widetilde{f}_n^{(i)}(x)$:

$$\sup_{x\in R} \big|\widetilde{f}_n^{(i)}(x)\big| \le 3C_f. \qquad (15)$$

Bhattacharya's paper [3] also implies that we can choose a constant C, so that for all $0 \le i \le m$ we have

$$\sup_{x\in R} \big|f^{(i)}(x) - f_n^{(i)}(x)\big| < Ch_n. \qquad (16)$$

With (15) and (16) in mind, (14) implies

$$I_2 \le \sum_{i=0}^{m} (3C_f)^{\alpha_i'} |s_n|^{\alpha_i} \int_{-s_n}^{s_n} |\widetilde{f}_n(x)|^{\alpha_{0i}} \big|f^{(i)}(x) - f_n^{(i)}(x)\big|\, dx \le$$

$$\le const \cdot h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i} \int_{-s_n}^{s_n} |\widetilde{f}_n(x)|^{\alpha_{0i}}\, dx, \qquad (17)$$

where α_i' denotes the sum of all positive orders of α_{ki} in (14).

Some of the orders of α_{0i} are positive while the others are negative. But in any case

$$|\widetilde{f}_n(x)|^{\alpha_{0i}} \leq C \cdot V(\alpha_{oi}, s_n).$$

Therefore (17) implies

$$I_2 \leq const \cdot h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1} V(\alpha_{oi}, s_n).$$

It is evident that if s_n and h_n are taken so that

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1} V(\alpha_{0i}; s_n) \to 0 \quad \text{for} \quad n \to \infty,$$

then $I_2 \to 0$ as $n \to \infty$. □

Proof of Lemma 3.2. For the proof we use the technique of paper [13].

Let W_m be the Sobolev space of functions from $L_2(R)$ with continuous derivatives up to the mth order inclusive with the norm

$$\|g\|_m = \sqrt{\sum_{j=0}^{m} \int_{-\infty}^{\infty} \left|g^{(j)}(x)\right|^2 dx}\,.$$

The space W_m has a scalar product

$$(g_1, g_2)_m = \sum_{j=0}^{m} \int_{-\infty}^{\infty} g_1^{(j)}(x) g_2^{(j)}(x)\, dx.$$

Let

$$r_n(m) = \|\widehat{f}_n - f_n\|_m^2$$

and

$$Y_i = Y_i(x) = \frac{1}{n}\left\{\frac{1}{h_n} K\Big(\frac{x - X_i}{h_n}\Big) - f_n(x)\right\}.$$

Then

$$\sum_{i=1}^{n} Y_i(x) = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{h_n} K\Big(\frac{x - X_i}{h_n}\Big) - f_n(x)\right\} = \widehat{f}_n(x) - f_n(x).$$

Hence

$$r_n(m) = \Big\| \sum_{i=1}^{n} Y_i(x) \Big\|_m^2. \tag{18}$$

Estimate the function

$$g_i = g_i(x) = \frac{1}{nh_n} K\Big(\frac{x - X_i}{h_n}\Big)$$

through the norm $\|\cdot\|_m$ for every $i = 1, \ldots, n$. We have

$$\|g_i\|_m^2 = \sum_{j=0}^{m} \frac{1}{n^2} \int_{-\infty}^{\infty} \Big(\frac{1}{h_n^{j+1}} K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\Big)^2 dx =$$

$$= \frac{1}{n^2} \sum_{j=0}^{m} \frac{1}{h_n^{2j+1}} \int_{-\infty}^{\infty} \Big(K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\Big)^2 d\,\frac{x - X_i}{h_n} \le$$

$$\le \frac{1}{n^2 h_n^{2m+1}} \sum_{j=0}^{m} \int_{-\infty}^{\infty} \big(K^{(j)}(u)\big)^2 du.$$

Thus

$$\|g_i\|_m \le \frac{1}{nh_n^{m+1/2}} \|K\|_m \stackrel{def}{=} A_n. \tag{19}$$

According to $(\boldsymbol{k3})$, $(\boldsymbol{k4})$, $\|K\|_m$ is finite. From (19) we have

$$\|Y_i\|_m \le \|g_i\|_m + E\|g_i\|_m \le 2A_n. \tag{20}$$

In order to estimate $r_n(m)$, we apply McDiarmid's inequality, which for convenience will be stated below.

McDiarmid's inequality. Let $L(y_1, \ldots, y_k)$ be a real function such that for each $i = 1, \ldots, m$ and some c_i, the supremum in $y_1, \ldots, y_k, y$, of the difference

$$\Big| L(y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_k) - L(y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_k) \Big| \le c_i.$$

Further, if $Y_1, \ldots, Y_k$ are independent random variables taking values in the domain of the function $L(y_1, \ldots, y_k)$, then for every $t > 0$,

$$P\Big\{ \big| L(Y_1, \ldots, Y_k) - EL(Y_1, \ldots, Y_k) \big| \ge t \Big\} \le 2e^{-\frac{2t^2}{\sum_{i=1}^{k} c_i^2}}.$$

We will apply McDiarmid's inequality to the expression

$$L(Y_1, \dots, Y_n) = \Big\| \sum_{i=1}^{n} Y_i \Big\|_m$$

and $c_i = 4A_n$ for $i = 1, \dots, n$. For any $t > 0$, taking into account (20), we have

$$P\left\{ \left| \Big\| \sum_{i=1}^{n} Y_i \Big\|_m - E\Big\| \sum_{i=1}^{n} Y_i \Big\|_m \right| \geq t \right\} \leq 2\exp\left\{ -\frac{t^2 n h_n^{2m+1}}{2\|K\|_m^2} \right\}. \tag{21}$$

Substitute

$$t = \frac{2\|K\|_m \sqrt{\log n}}{\sqrt{n h_n^{2m+1}}}$$

in (21) and apply the Borel–Cantelli lemma. Then, with probability 1, we have

$$\Big\| \sum_{i=1}^{n} Y_i \Big\|_m = E\Big\| \sum_{i=1}^{n} Y_i \Big\|_m + O\left(\frac{\sqrt{\log n}}{\sqrt{n h_n^{2m+1}}} \right). \tag{22}$$

Now estimate $\| \sum_{i=1}^{n} Y_i \|_m^2$. In order to do this, apply Jensen's inequality

$$\left(E\Big\| \sum_{i=1}^{n} Y_i \Big\|_m \right)^2 \leq E\Big\| \sum_{i=1}^{n} Y_i \Big\|_m^2 = \sum_{i=1}^{n} \sum_{j=0}^{m} \int_{-\infty}^{\infty} \left(Y_i^{(j)}(x) \right)^2 dx \leq$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=0}^{m} \int_{-\infty}^{\infty} E\left\{ \frac{1}{h_n^{j+1}} K^{(j)}\left(\frac{x - X_i}{h_n} \right) - f_n^{(j)}(x) \right\}^2 dx \leq$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=0}^{m} \left\{ \int_{-\infty}^{\infty} E\left(\frac{1}{h_n^{j+1}} K^{(j)}\left(\frac{x - X_i}{h_n} \right) \right)^2 dx \right\}^2 \leq$$

$$\leq \frac{1}{n^2 h_n^{2m+2}} \sum_{i=1}^{n} \sum_{j=0}^{m} \left\{ \int_{-\infty}^{\infty} E\left(K^{(j)}\left(\frac{x - X_i}{h_n} \right) \right)^2 dx \right\} =$$

$$= \frac{1}{n^2 h_n^{2m+2}} \sum_{i=1}^{n} \sum_{j=0}^{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (K^{(j)})^2 \left(\frac{x - y}{h_n} \right) f(y)\, dy\, dx \leq$$

$$\leq \frac{C_f}{n h_n^{2m+1}} \|K\|_m^2. \tag{23}$$

It follows from (18), (22) and (23) that $r_n(m) = O(\frac{\log n}{nh_n^{2m+1}})$ almost everywhere. □

Proof of Lemma 3.3. We have

$$R_n = \frac{1}{2}\sum_{i,j=0}^{m}\int_{-s_n}^{s_n} \varphi_{(ij)}(\widetilde{y}_m(x))\big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big)\big(\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big)\,dx, \quad (24)$$

where $\widetilde{y}_m(x)$ is a point on the line connecting the points

$$\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big) \text{ and } \big(x, \widehat{f}_n(x), \ldots, \widehat{f}_n^{(m)}(x)\big).$$

Estimate R_n. It follows from (8) that under the condition

$$\frac{\log\log n}{nh_n^{2m+1}} \to 0, \quad n \to \infty$$

we have

$$-C_f \le \widehat{f}_n^{(i)}(x) \le C_f, \quad i = 0, 1, \ldots, m. \quad (25)$$

Therefore by virtue of $(\boldsymbol{f2})$, $(\boldsymbol{\varphi 3})$, (24) leads to

$$|R_n| \le \frac{1}{2}C_\varphi\sum_{i,j=0}^{m}\int_{-s_n}^{s_n} |x|^{\beta_{ij}}|\widetilde{y}_0(x)|^{\beta_{0ij}}\cdots|\widetilde{y}_m(x)|^{\beta_{mij}}\times$$
$$\times\,\big|\,\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big|\,\big|\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big|\,dx.$$

This implies

$$|R_n| \le C\sum_{i,j=0}^{m} C_f^{\beta'_{ij}}|s_n|^{\beta_{ij}}\times$$
$$\times\int_{-s_n}^{s_n} |\widetilde{y}_0(x)|^{\beta_{0ij}}\big|\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big|\,\big|\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big|\,dx, \quad (26)$$

where β'_{ij} denotes a sum of positive numbers orders: $\beta'_{ij} = \beta_{1ij} + \cdots + \beta_{mij}$. Some of the numbers β_{0ij} are positive, others are negative. In any case

$$|\widetilde{f}_n(x)|^{\beta_{0ij}} \le C \cdot V(\beta_{0ij}, s_n).$$

Then (26) results in

$$|R_n| \leq const \cdot \sum_{i=0}^{m} \sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}, s_n) r_n^{ij}(x). \tag{27}$$

But by virtue of Lemma 3.2 we have $r_n^{ij} = O(\frac{\log n}{nh_n^{2m+1}})$, therefore (27) gives

$$R_n = O\Big(\frac{d_m(s_n)\log n}{nh_n^{2m+1}}\Big),$$

where

$$d_m(s_n) = \sum_{i=0}^{m} \sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}; s_n). \qquad \square$$

Proof of Theorem 3.1. Note that

$$\big|I(f) - I(\widehat{f}_n, s_n)\big| \leq \big|I(f) - I(f_n, s_n)\big| + \big|I(\widehat{f}_n, s_n) - I(f_n, s_n)\big|, \tag{28}$$

where

$$I(f_n, s_n) = \int_{-s_n}^{s_n} \varphi\big(x, f_n(x), f_n'(x), \ldots, f_n^{(m)}(x)\big)\, dx.$$

Furthermore

$$\big|I(f) - I(f_n, s_n)\big| \leq \Big|\int_{|x|\geq s_n} \varphi\big(x, f(x), f'(x), \ldots, f^{(m)}(x)\big)\, dx\Big| +$$

$$+ \Big|\int_{-s_n}^{s_n} \Big\{\varphi\big(x, f(x), \ldots, f^{(m)}(x)\big) - \varphi\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\Big\}\, dx\Big| \doteqdot I_1 + I_2.$$

Condition **(φ3)** implies that $I_1 \to 0$ as $s_n \to \infty$. In order to establish the convergence $I_2 \to 0$, we verify the fact that under the conditions of the theorem Lemma 3.1 is true. Suppose $\theta \geq 0$. Then

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1} V(\alpha_{0i}; s_n) \leq$$

$$\leq const \cdot m \cdot h_n \cdot s_n^{\tau} = const \cdot m \cdot h_n^{\frac{1+\rho-\tau}{1+\rho}} \longrightarrow 0 \text{ as } n \to \infty.$$

Now assume that $\theta < 0$. Then

$$h_n \cdot \sum_{i=0}^{m} |s_n|^{\alpha_i+1} V(\alpha_{0i}; s_n) \le$$

$$\le \sum_{i=0}^{m} H^{\alpha_i+1+|\alpha_{oi}|}(s_n) \le m \cdot H^{\tau}(s_n) = m h_n^{\frac{1+\rho-\tau}{\rho+1}} \longrightarrow 0 \text{ as } n \to \infty.$$

Hence the conditions of Lemma 3.1 can be satisfied. Thus $I_2 \to 0$.

In (28) we have

$$I(\widehat{f}_n, s_n) - I(f_n, s_n) = S_n(h_n) + R_n =$$

$$= \sum_{i=0}^{m} \int_{-s_n}^{s_n} \varphi_{(i)}\big(x, f_n(x), \dots, f_n^{(m)}(x)\big)\big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big)\,dx + R_n, \quad (29)$$

with

$$R_n = \frac{1}{2} \sum_{i,j=0}^{m} \int_{-s_n}^{s_n} \varphi_{(ij)}(\widetilde{y}_m(x))\big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big)\big(\widehat{f}_n^{(j)}(x) - f_n^{(j)}(x)\big)\,dx, \quad (30)$$

where $\widetilde{y}_m(x)$ is a point on the line connecting the points

$$\big(x, f_n(x), \dots, f_n^{(m)}(x)\big) \text{ and } \big(x, \widehat{f}_n(x), \dots, \widehat{f}_n^{(m)}(x)\big).$$

By virtue of Lemma 3.3

$$R_n = O\Big(\frac{d_m(s_n)\log n}{n h_n^{2m+1}}\Big),$$

where

$$d_m(s_n) = \sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}; s_n).$$

Let $\theta \ge 0$. Then

$$\sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}; s_n) \le const \cdot m \cdot s_n^{\sigma}.$$

Consequently,

$$\frac{d_m(s_n)\log n}{nh_n^{2m+1}} \le const \cdot \frac{\log n}{nh_n^{2m+1+\frac{\sigma}{1+\rho}}} \to 0 \ \text{ as } \ n \to \infty.$$

Let $\theta < 0$. Then

$$\sum_{i=0}^{m}\sum_{j=0}^{m} |s_n|^{\beta_{ij}} V(\beta_{0ij}; s_n) \le \sum_{i,j=0}^{m} H^{\beta_{ij}+|\beta_{0ij}|}(s_n) \le m^2 \cdot H^{\sigma}(s_n) = m^2 h_n^{-\frac{\sigma}{\rho+1}}.$$

Therefore

$$\frac{\log n}{nh_n^{2m+1}} m^2 h_n^{-\frac{\sigma}{\rho+1}} = \frac{m^2 \log n}{nh_n^{2m+1+\frac{\sigma}{\rho+1}}} \to 0 \ \text{ as; } \ n \to \infty.$$

Consequently, $R_n \to 0$ with probability 1 as $n \to \infty$.

Now estimate the main summand

$$S_n(h_n) = \sum_{i=0}^{m} \int_{-s_n}^{s_n} \varphi_{(i)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big)\, dx. \qquad (31)$$

Let

$$Z_i(h_n) \doteqdot \sum_{j=0}^{m} \int_{-s_n}^{s_n} \varphi_{(j)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big) \frac{1}{h_n^{j+1}} K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\, dx.$$

$S_n(h_n)$ can be represented as a sum of independent random variables

$$S_n(h_n) = \frac{1}{n} \sum_{i=1}^{n} \big\{Z_i(h_n) - EZ_i(h_n)\big\}.$$

$[-\mathbb{k}, \mathbb{k}]$ is the smallest interval containing the support of the function $K(x)$ (the existence of such an interval follows from $\boldsymbol{(k4)}$. Bearing condition $\boldsymbol{(\varphi 3)}$ in mind, we write

$$|Z_i(h_n)| \le$$

$$\le C_{\varphi} \sum_{j=0}^{m} \int_{-s_n}^{s_n} |x|^{\alpha_j} |f_n(x)|^{\alpha_{0j}} \cdots |f_n^{(m)}(x)|^{\alpha_{mj}} \frac{1}{h_n^{j+1}} K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\, dx.$$

Therefore

$$|Z_i(h_n)| \le const \cdot \sum_{j=0}^{m} |s_n|_j^{\alpha} V(\alpha_{0j}; s_n) \int_{-s_n}^{s_n} \frac{1}{h_n^{j+1}} K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\, dx. \qquad (32)$$

Note that for a sufficiently large $n$we have $s_n > k$. Hence from (32) we have

$$|Z_i(h_n)| \le const \cdot \sum_{j=0}^{m} \frac{1}{h_n^{j+1}} \int_{-\Bbbk}^{\Bbbk} K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\, dx \le B h_n^{-m}.$$

For a sufficiently large n and some B.

Now we apply McDiarmid's inequality to the value

$$S_n(h_n) = \frac{1}{n} \sum_{i=1}^{n} \big\{ Z_i(h_n) - E Z_i(h_n) \big\}.$$

We have

$$P\big\{|S_n(h_n)| > t\big\} \le 2 \exp\Big\{ - \frac{n t^2 h_n^{2m}}{2B^2} \Big\}.$$

Take

$$t = \frac{2B\sqrt{\log n}}{\sqrt{n}\, h_n^m}.$$

We obtain

$$P\Big\{ |S_n(h_n)| > \frac{2B\sqrt{\log n}}{\sqrt{n}\, h_n^m} \Big\} \le 2 \exp\{-2 \log n\}.$$

According to the Borel–Cantelli lemma, with probability 1, we have

$$S_n(h_n) = O\Bigg(\sqrt{\frac{\log n}{n h_n^{2m}}} \Bigg).$$

Note that condition (B) implies the convergence

$$\frac{\log n}{n h_n^{2m}} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $S_n(h_n) \to 0$ with probability 1 as $n \to \infty$. This completes the proof of the theorem. □

Proof of Theorem 3.2. Remember the representation

$$I(\widehat{f}_n, s_n) - I(f_n, s_n) = S_n(h_n) + R_n =$$
$$= \sum_{i=0}^{m} \int_{-s_n}^{s_n} \varphi_{(i)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\big(\widehat{f}_n^{(i)}(x) - f_n^{(i)}(x)\big)\, dx + R_n.$$

If

$$Z_i(h_n) \doteqdot \sum_{j=0}^{m} \int_{-s_n}^{s_n} \varphi_{(j)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\, \frac{1}{h_n^{j+1}}\, K^{(j)}\Big(\frac{x - X_i}{h_n}\Big)\, dx,$$

then$S_n(h_n)$ can be represented as a sum of independent random variables

$$S_n(h_n) = \frac{1}{n} \sum_{i=1}^{n} \big\{Z_i(h_n) - EZ_i(h_n)\big\}.$$

Our aim is to find moments of the value $S_n(h_n)$.

We have

$$EZ_i(h_n) =$$
$$= \sum_{j=0}^{m} \int_{-\infty}^{\infty} \Big\{ \int_{-s_n}^{s_n} \varphi_{(j)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\, \frac{1}{h_n^{j+1}}\, K^{(j)}\Big(\frac{x-y}{h_n}\Big)\, dx \Big\} f(y)\, dy =$$
$$= \sum_{j=0}^{m} \int_{-\infty}^{\infty} \Bigg\{ \int_{\frac{-y-s_n}{h_n}}^{\frac{-y+s_n}{h_n}} \varphi_{(j)}\Big(y + uh_n, f_n(y+uh_n), \ldots, f_n^{(m)}(y+uh_n)\Big) \times$$
$$\times \frac{1}{h_n^{j}}\, K^{(j)}(u)\, du \Bigg\} f(y)\, dy. \qquad (33)$$

As n increases, by virtue of property $\boldsymbol{(k4)}$, in the inner integral of formula (33) the integration limits are $-\Bbbk$ and $\Bbbk$. Apply the integration by parts formula several times. Then we can write

$$\int_{-\Bbbk}^{\Bbbk} \varphi_{(j)}\Big(y+uh_n, f_n(y+uh_n), \ldots, f_n^{(m)}(y+uh_n)\Big)\,\frac{1}{h_n^j}\,K^{(j)}(u)\,du =$$

$$= (-1)^j \int_{-\Bbbk}^{\Bbbk} \frac{1}{h_n^j}\,\frac{d^j}{du^j}\,\varphi_{(j)}\Big(y+uh_n, f_n(y+uh_n), \ldots, f_n^{(m)}(y+uh_n)\Big)K(u)\,du.$$

For $n \to \infty$, $h_n \downarrow 0$ and

$$\frac{1}{h_n^j}\,\frac{d^j}{du^j}\,\varphi_{(j)}\Big(y+uh_n, f_n(y+uh_n), \ldots, f_n^{(m)}(y+uh_n)\Big) \longrightarrow$$

$$\longrightarrow \frac{d^j}{dy^j}\,\varphi_{(j)}\big(y, f(y), \ldots, f^{(m)}(y)\big) \overset{def}{=} \varphi_{(j)}^{(j)}(y).$$

Denote

$$q_m(y) = \sum_{j=0}^{m} (-1)^j \varphi_{(j)}^{(j)}(y).$$

Then we see that as $n \to \infty$

$$EZ_i(h_n) \longrightarrow \int_{-\infty}^{\infty} \Big\{ \sum_{j=0}^{m} (-1)^j \varphi_{(j)}^{(j)}(y) \Big\} f(y)\,dy =$$

$$= \int_{-\infty}^{\infty} q_m(y) f(y)\,dy = Eq_m(X).$$

Now, let $0 \le j, v \le m$. Consider the value

$$\mu_{j,v}(y) = \int_{-s_n}^{s_n}\int_{-s_n}^{s_n} \varphi_{(j)}\big(x, f_n(x), \ldots, f_n^{(m)}(x)\big)\varphi_{(v)}\big(z, f_n(z), \ldots, f_n^{(m)}(z)\big)\times$$

$$\times \frac{1}{h_n^{2+j+v}}\,K^{(j)}\Big(\frac{x-y}{h_n}\Big)K^{(v)}\Big(\frac{z-y}{h_n}\Big)\,dx\,dz.$$

For a large enough n

$$\mu_{j,v}(y) = \int_{-\Bbbk}^{\Bbbk}\int_{-\Bbbk}^{\Bbbk} \varphi_{(j)}\Big(y+uh_n, f_n(y+uh_n),\dots,f_n^{(m)}(y+uh_n)\Big)\times$$
$$\times\,\varphi_{(v)}\Big(y+\tau h_n, f_n(y+\tau h_n),\dots,f_n^{(m)}(y+\tau h_n)\Big)\times$$
$$\times\,\frac{1}{h_n^{j+v}}\,K^{(j)}(u)K^{(v)}(\tau)\,du\,d\tau.$$

Hence we conclude that

$$E\mu_{j,v}(y) \longrightarrow (-1)^{j+v}\int_{-\infty}^{\infty}\varphi_{(j)}^{(j)}(y)\varphi_{(v)}^{(v)}(y)f(y)\,dy.$$

Respectively,

$$EZ_i^2(h_n) = \sum_{j=0}^{m}\sum_{v=0}^{m}\int_{-\infty}^{\infty}\mu_{j,v}(y)f(y)\,dy.$$

So we can conclude that as $n\to\infty$

$$EZ_i^2(h_n) \longrightarrow \sum_{j=0}^{m}\sum_{v=0}^{m}(-1)^{j+v}\int_{-\infty}^{\infty}\varphi_{(j)}^{(j)}(y)\varphi_{(v)}^{(v)}(y)f(y)\,dy =$$
$$= \int_{-\infty}^{\infty} q_m^2(y)f(y)\,dy = Eq_m^2(X).$$

Quite similarly we can show that as $n\to\infty$

$$EZ_i^4(h_n) \to \int_{-\infty}^{\infty} q_m^4(y)f(y)\,dy = Eq_m^4(X).$$

After these computations we can arrive at the conclusion that under the above-stated conditions as $n\to\infty$, $h_n\to 0$

$$n\,\mathrm{Var}(S_n(h_n)) = \mathrm{Var}(Z_i(h_n)) \longrightarrow \mathrm{Var}(q_m(f(X))) \overset{def}{=} \sigma^2(f) < \infty$$

and $EZ_i^4(h_n)\to Eq_m^4(X)<\infty$.

A reference to Lyapunov's central limit theorem completes the proof. □

5 Some Examples

(i)

Consider the integral functional

$$I_{F,k}(f) = \int_{-\infty}^{\infty} \frac{(f'(x))^{2k}}{f(x)}\,dx, \quad k \geq 1.$$

We will estimate this integral by means of

$$I_{F,k}(\widehat{f}_n, s_n) = \int_{-s_n}^{s_n} \frac{(\widehat{f}_n{}'(x))^{2k}}{\widehat{f}_n(x)}\,dx, \quad k \geq 1,$$

where

$$\widehat{f}_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\Big(\frac{x - X_i}{h_n}\Big), \quad \widehat{f}_n{}'(x) = \frac{1}{nh_n^2}\sum_{i=1}^{n} K'\Big(\frac{x - X_i}{h_n}\Big).$$

In this case $m = 1$,

$$\varphi(x, x_0, x_1) = x_0^{-1}x_1^{2k}, \quad \frac{\partial\varphi}{\partial x_0} = -x_0^{-2}x_1^{2k}, \quad \frac{\partial\varphi}{\partial x_1} = 2kx_0^{-1}x_1^{2k-1},$$
$$\frac{\partial^2\varphi}{\partial x_1^2} = 2k(2k-1)x_0^{-1}x_1^{2k-2}, \quad \frac{\partial^2\varphi}{\partial x_0^2} = 2x_0^{-3}x_1^{2k}, \quad \frac{\partial^2\varphi}{\partial x_0\partial x_1} = -2kx_0^{-2}x_1^{2k-1},$$
$$C_\varphi = 2k(2k-1), \quad \alpha_0 = \alpha_1 = = \beta_{00} = \beta_{01} = \beta_{10} = \beta_{11} = 0, \quad \alpha_{00} = -2,$$
$$\alpha_{10} = 2k, \quad \alpha_{01} = -1, \quad \alpha_{11} = 2k-1, \quad \theta < 0,$$
$$\beta_{000} = -3, \quad \beta_{100} = 2k, \quad \beta_{001} = -2, \quad \beta_{101} = 2k-1, \quad \beta_{010} = -2,$$
$$\beta_{110} = 2k-1, \quad \beta_{011} = -1, \quad \beta_{111} = 2k-2,$$
$$\sigma = 3, \quad \tau = 3, \quad \rho = 3, \quad s_n = H^{-1}\Big(h_n^{-\frac{1}{4}}\Big).$$

For example, we can take $h_n = n^{-\frac{9}{40}}$ as h_n then $s_n = H^{-1}(n^{\frac{9}{160}})$. In, particular, examples we often have $H(x) = 3e^{x^2}$. Then $s_n = \sqrt{\log\frac{1}{3}\,n^{\frac{9}{160}}}$. Note that this sequence diverges rather slowly.

However, under conditions **(*f*1)**–**(*f*4)**, **(*k*1)**–**(*k*4)** Theorem **??** is true and we have $I_{F,k}(\widehat{f}_n, s_n) \to I_{F,k}(f)$ with probability 1. If $k = 1$, we have Fisher's information function.

In the central limit theorem for the Fisher information integral $I_F(f)$ we have

$$\sigma^2(f) = \text{Var}\, \frac{(f'(X))^2 + 2f''(X)f(X)}{(f(X))^2}\,.$$

(ii)

Consider the integral

$$I_{S,\delta}(f) = \int_{-\infty}^{\infty} f^{\delta}(x) \log f(x)\, x, \quad 0 < \delta \leq 1,$$

which will be estimated by means of

$$I_{S,\delta}(\widehat{f}_n, s_n) = \int_{-s_n}^{s_n} \widehat{f}_n^{\,\delta}(x) \log \widehat{f}_n(x)\, dx,$$

where

$$\widehat{f}_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\Big(\frac{x - X_i}{h_n}\Big).$$

In this case $m = 0$ and

$$\varphi(x, x_0, \ldots, x_m) = x_0^{\delta} \log x_0.$$

Then

$$\frac{d\varphi}{dx_0} = x_0^{\delta-1}(\log x_0 + 1) < e x_0^{2\delta-1},$$

$$\frac{d^2\varphi}{dx_0^2} = x_0^{\delta-2}\big(\delta(\delta-1)\log x_0 + 2\delta - 1\big) < e^{2\delta-1} x_0^{\delta^2-2},$$

$$C_{\varphi} = e, \quad \alpha_0 = 0, \quad \alpha_{00} = 2\delta - 1, \quad \beta_{00} = 0, \quad \beta_{000} = \delta^2 - 2, \quad \theta < 0,$$

$$\tau = 2 - 2\delta, \quad \sigma = 2 - \delta^2, \quad \rho = 2 - \delta^2, \quad s_n = H^{-1}(h_n^{-\frac{1}{3-\delta^2}}).$$

For example, one can take $h_n = n^{-\frac{5}{12}}$ for h_n. Then the conditions of Theorem 3.2 are satisfied. Under conditions **(*f*1)–(*f*4)**, **(*k*1)–(*k*4)** we have $I_F(\widehat{f}_n, s_n) \to I_F(f)$ with probability 1. For $k = 1$ we have Shannon's entropy.

In the central limit theorem we have

$$\sigma^2(f) = \operatorname{Var} \log f(X)$$

for Shannon's entropy integral.

(iii)

Consider the integral

$$I(f;\gamma) = \int_{-\infty}^{\infty} x^{\gamma}(f'(x))^2\,dx, \quad \gamma \geq 0.$$

The estimator of this interval has the form

$$I_n(\widehat{f}_n;\gamma) = \int_{-s_n}^{s_n} x^{\gamma}\left(\widehat{f}\,'_n(x)\right)^2, \quad \widehat{f}_n{}'(x) = \frac{1}{nh_n^2}\sum_{i=1}^{n} K'\Big(\frac{x-X_i}{h_n}\Big).$$

We have $m = 1$,

$$\varphi(x, x_0, x_1) = x^{\gamma}x_1^2, \quad \frac{\partial\varphi}{\partial x_1} = 2x^{\gamma}x_1, \quad \frac{\partial^2\varphi}{\partial x_1^2} = 2x^{\gamma},$$
$$C_{\varphi} = 2, \ \ \alpha_0 = 0, \ \ \alpha_1 = \gamma, \ \ \alpha_{00} = 0, \ \ \alpha_{01} = 0, \ \ \alpha_{10} = 0, \ \ \alpha_{11} = 1,$$
$$\beta_{00} = 0, \ \ \beta_{01} = 0, \ \ \beta_{10} = 0, \ \ \beta_{11} = \gamma, \ \ \beta_{000} = 0,$$
$$\beta_{001} = 0, \ \ \beta_{010} = 0, \ \ \beta_{011} = 0, \ \ \beta_{100} = 0, \ \ \beta_{101} = 0,$$
$$\beta_{110}, \ \ \beta_{111} = 0, \ \ \theta > 0, \ \ \tau = 1+\gamma, \ \ \sigma = \gamma, \ \ \rho = 1+\gamma, \ \ s_n = h_n^{-\frac{1}{2+\gamma}}.$$

Condition $(\boldsymbol{f4})$ is redundant. One can take $h_n = n^{-\delta}$ as h_n, where $\frac{1}{5} < \delta < \frac{1}{4}$. By virtue of Theorem 3.1, we have $I_n(\widehat{f}_n;\gamma) \to I(f;\gamma)$ with probability 1.

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