Stochastic integral representations of Brownian functionals

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As it is well-known from Ito's calculus, stochastic integral from square integrable adapted process is square integrable martingale. The answer on the inverse question: is it possible to represent the square integrable martingale adapted to the natural filtration of Brownian motion, as the stochastic integral is given by well-known Clark formula (1971). In particular, let B_t ($t \in [0,T]$) be standard Brownian motion and $\Im_t$ is a natural filtration generated by this Brownian motion. If F is a square integrable $\Im_T$-measurable random variable, then there exist square integrable $\Im_t$-adapted random process φ_t such that $F = EF + \int_0^T \varphi_t dB_t$. On the other hand, finding of explicit expression for φ_t is very difficult problem. In this direction, it is known one general result, called Ocone-Clark formula (1984), according to which $\varphi_t = E(D_t F \mid \Im_t)$, where D_t is so called Malliavin stochastic derivative. But, on the one hand, here it is required the stochastically smoothness and on the other hand, even in case of smoothness, calculations of Malliavin derivative and conditional mathematical expectation are rather difficult.

The next step in this direction was taken by Ma, Protter and Martin (1998), they offered the concept of stochastic derivative and generalized stochastic integral for so called normal martingales class and generalized Clark's formula for functionals from the class $D_{2,1}^M$ (the functional $F = \sum_{n=0}^{\infty} I_n(f_n)$ belongs to the space $D_{2,1}^M$ if and only if $\sum_{n=1}^{\infty} nn! \| f_n \|^2_{L_2([0,T]^n)} < \infty$). We (Purtukhia, 2003) have introduced the space $D_{p,1}^M$, $1 < p < 2$ ($D_{p,1}^M$ the Banach space which is the closure of $D_{2,1}^M$ under the following norm $\| F \|_{p,1} := E(\| F \|_{L_p} + \| D_{\cdot} F \|_{L_2([0,T])})$) and extended the Ocone-Haussmann-Clark formula for functionals from this space. Absolutely different method for finding of φ_t was offered by Shyriaev, Yor and Graversen (2003, 2006), which was based on using of Ito's (generalized) formula and Levy's theorem for associated to F Levy's martingale $m_t = E(F \mid \Im_t)$. We (Purtukhia, Jaoshvili, 2009) introduced the new construction of stochastic derivative of Poisson functional and established the explicit expression for the integrand of Clark representation.

In the all cases described above F was stochastically smooth. We (with prof. O. Glonti, 2014) considered case when F is non stochastically smooth, but from associated with F Levy's martingale one can to select a stochastically smooth subsequence and in this case we gave the method for finding of integrand. Here we consider a different case, when functional represents the Lebesgue integral from stochastically non smooth square integrable process, with respect to time variable.

Theorem. The following stochastic integral representation is fulfilled

$$\int_0^T I_{\{B_t \le c\}} dt = \int_0^T \Phi\left(\frac{c}{\sqrt{t}}\right) dt - \int_0^T \left(\int_s^T \frac{1}{\sqrt{t-s}} \varphi\left(\frac{c - B_s}{\sqrt{t-s}}\right) dt \right) dB_s ,$$

where Φ (respectively φ) is the standard normal distribution function (respectively density), $c = const$.

Acknowledgement. The work is supported by Shota Rustaveli National Science Foundation Grant No FR/308/5-104/12.